ELSEVIER Linear Algebra and its Applications 306 (2000) 145–154
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Centrogonal matrices

Olaf Krafft*, Martin Schaefer

Institut für Statistik, RWTH Aachen, D-52056 Aachen, Germany

Received 7 December 1998; accepted 28 October 1999

Submitted by F. Uhlig

Abstract

A nonsingular $n \times n$-matrix $A = (a_{ij})$ is called centrogonal if $A^{-1} = (a_{n+1-i,n+1-j})$; it is called principally centrogonal if all leading principal submatrices of A are centrogonal. A characterization theorem for such matrices is proved. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Matrix rotation; Binomial sums

1. Introduction

Trying to find the inverse of a special type of matrices, an example of which is the matrix

$$A = \begin{pmatrix} 1 & -4 & 6 & -4 \\ 4 & -15 & 20 & -10 \\ 10 & -36 & 45 & -20 \\ 20 & -70 & 84 & -35 \end{pmatrix}, \tag{1.1}$$

we observed a phenomenon which seemed to be interesting enough to be studied in more detail. The inverse of A is just the matrix A rotated around its center, i.e.

$$A^{-1} = \begin{pmatrix} -35 & 84 & -70 & 20 \\ -20 & 45 & -36 & 10 \\ -10 & 20 & -15 & 4 \\ -4 & 6 & -4 & 1 \end{pmatrix}.$$

* Corresponding author. Tel.: +49-241-80-4610; fax: +49-241-8888-130.

PII: S 0 0 2 4 - 3 7 9 5 (9 9) 0 0 2 4 7 - 5

In analogy to the fact that the result of transposing a matrix is called the transpose, we call the result of rotating a matrix the rotate. Thus for a square matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ the rotate A^{R} is defined by $A^{\mathrm{R}} = (a_{n+1-i,n+1-j})_{1\leqslant i,j\leqslant n}$. The rotation operator has already been used in connection with the concept of symmetry: A is called centrosymmetric if $A = A^{\mathrm{R}}$. Properties of the determinant of centrosymmetric matrices have been obtained by Muir [7] and Aitken [1]. Good [4] shows how to find the inverse of centrosymmetric matrices and Weaver [8] presents results on their eigenvalues and eigenvectors. Generalizations of centrosymmetry are discussed by Hill and Waters [6]. The formal analogy to orthogonality, i.e. the gist of the example above, does not seem to have been considered yet.

Definition 1. A nonsingular matrix A is called centrogonal if $A^{-1} = A^{\mathrm{R}}$.

A closer look at the example reveals an even more surprising feature: Not only A is centrogonal but also all its leading principal submatrices $A_m = (a_{ij})_{1\leqslant i,j\leqslant m}$, $1 \leqslant m \leqslant 3$. The supplement to Definition 1 thus becomes the following.

Definition 2. A nonsingular $n \times n$ matrix $A = (a_{ij})$ is called principally centrogonal if all its leading principal submatrices $A_m = (a_{ij})_{1\leqslant i,j\leqslant m}$, $1 \leqslant m \leqslant n$, are centrogonal.

In Section 2, we give some elementary properties of centrogonal matrices. A characterization theorem shows how to construct such matrices. It will turn out that there is a close connection to involutions. The concept of principal centrogonality, however, adds some additional features. These allow a – at least to us – surprisingly simple characterization (Theorem 2). In Section 3, the introductory example will be studied in more generality. This section could also be considered as an extended exercise for working with binomial identities. Closed forms for sums with three or more binomial coefficients are, according to Graham et al. [5, p. 171] rarities. As a by-product, two such rarities will be obtained here.

2. Centrogonal matrices

All matrices considered in this paper are real. In addition to the notations introduced in Section 1, we denote by I_n the unit matrix of order n; J_n is the $n \times n$ exchange matrix, i.e.,

$$J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

(The index n will be omitted if it is obvious from the context.) For a vector $\mathbf{a} = (a_1, \ldots, a_n)^{\mathrm{T}}$ the triangular $n \times n$ matrix $B(\mathbf{a})$ is defined by

$$B(\mathbf{a}) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & a_1 \end{pmatrix}.$$

Clearly, for the rotate A^{R} of A it holds that $A^{\mathrm{R}} = JAJ$. Using $J^2 = I$ and $J = J^{\mathrm{T}}$, this immediately implies that the R-operator has the following properties

$$(A^{\mathrm{T}})^{\mathrm{R}} = (A^{\mathrm{R}})^{\mathrm{T}}, \tag{2.1}$$

$$(A^{\mathrm{R}})^{\mathrm{R}} = A, \tag{2.2}$$

$$(A^{-1})^{\mathrm{R}} = (A^{\mathrm{R}})^{-1}, \tag{2.3}$$

$$(AB)^{\mathrm{R}} = A^{\mathrm{R}} B^{\mathrm{R}}. \tag{2.4}$$

Equally simple is that a nonsingular matrix A is centrogonal if and only if AJ is an involution, i.e. $(AJ)^2 = I$. The following theorem shows how to generate all centrogonal matrices.

Theorem 1. *$A \in \mathbb{R}^{n\times n}$ is centrogonal if and only if there exist an $\eta \in \{-1, 1\}$, a nonsingular matrix $B \in \mathbb{R}^{n\times n}$ and a symmetric $n \times n$ permutation matrix Π such that $A = \eta B^{-1} \Pi B J$.*

Proof. Since $B^{-1}\Pi B$ is an involution, the if-part is obvious. Let now A be centrogonal; thus $C = AJ$ is an involution. Hence all eigenvalues of C are 1 or -1. It follows that the Jordan canonical form of C is $C = T^{-1}\Lambda T$, where Λ is a *diagonal* matrix and T a nonsingular (possibly complex) matrix. Since all diagonal elements of Λ are 1 or -1, T can be chosen as a real matrix. Let r denote the number of diagonal elements of Λ which are equal to -1 and let $2r \leqslant n$. Then

$$\Pi = \begin{pmatrix} J_{2r} & 0 \\ 0 & I_{n-2r} \end{pmatrix}$$

is a symmetric permutation matrix with characteristic polynomial $\varphi(\lambda) = (-1)^{n-2r} (\lambda - 1)^{n-r}(\lambda + 1)^r$. Hence Π is similar to Λ and there exists an orthogonal matrix U such that $\Pi = U^{\mathrm{T}}\Lambda U$. Putting $B = U^{\mathrm{T}}T$ one gets

$$B^{-1}\Pi B J = T^{-1}UU^{\mathrm{T}}\Lambda UU^{\mathrm{T}}TJ = CJ = A.$$

If $2r > n$ the same arguments as above can be applied to $-A$. $\square$

Remark 1. Theorem 1 shows in particular that the matrix $\eta A^{-1}A^{\mathrm{R}}$ is centrogonal if $\eta \in \{-1, 1\}$ and A is nonsingular. A further specialization yields that $\eta B^{-1}B^{\mathrm{T}}$ is centrogonal if $\eta \in \{-1, 1\}$ and B is nonsingular and persymmetric, i.e. $B = JB^{\mathrm{T}}J$.

Since Toeplitz matrices are persymmetric, cf. e.g. [3, p. 184], $\eta B^{-1}B^{\mathrm{T}}$ is centrogonal if B is a nonsingular Toeplitz matrix.

Example 1. Let $D=(d_{ij})$ be defined by

$$d_{ij}=(-1)^{j+1}\binom{n-i}{j-1},\quad 1\leqslant i,j\leqslant n,$$

where

$$\binom{m}{k}=0\quad \text{if}\ \ k<0\ \ \text{or}\ \ k>m.$$

Then, with the help of formula (5.24) in [5], one gets

$$\begin{aligned}\sum_{\varrho=1}^{n} d_{i\varrho}d_{\varrho j}&=(-1)^{j}\sum_{\varrho=1}^{n}(-1)^{\varrho}\binom{n-i}{\varrho-1}\binom{n-\varrho}{j-1}\\&=(-1)^{n+i+j}\sum_{\sigma=-1}^{n-2}(-1)^{\sigma}\binom{n-i}{\sigma+1}\binom{\sigma+i}{j-1}\\&=(-1)^{j+1}\binom{i-1}{n-j},\end{aligned}$$

thus $D^2=(-1)^{n+1}D^{\mathrm{R}}$, or, since D is nonsingular, $D=(-1)^{n+1}D^{-1}D^{\mathrm{R}}$. From Remark 1 it follows that D is centrogonal and this implies that $D^{-1}=(d'_{ij})$ is given by

$$d'_{ij}=(-1)^{n+j}\binom{i-1}{n-j},\quad 1\leqslant i,j\leqslant n.$$

A similar example can be found in [2, p. 19].

Principal centrogonality is, of course, a strong condition. Nevertheless, it might be surprising that by this property a matrix is completely determined by its first row.

Theorem 2. *Let $A_n=(a_{ij})$ be a nonsingular $n\times n$ matrix, $\mathbf{a}=(a_{11},a_{12},\dots,a_{1n})^{\mathrm{T}}$ and $B=B(\mathbf{a})$. Then A_n is principally centrogonal if and only if $A_n=a_{11}B^{-1}B^{\mathrm{T}}$ and $a_{11}\in\{-1,1\}$.*

Proof. Let $A_n=a_{11}B^{-1}B^{\mathrm{T}}$ and $a_{11}\in\{-1,1\}$. The special structure of B immediately gives $B^{\mathrm{R}}=B^{\mathrm{T}}$. Thus, by Remark 1, A_n is centrogonal. Let $\hat{A}_m=(a_{ij})_{1\leqslant i,j\leqslant m}$, $m<n$. Then $\hat{A}_m=[I_m,0]A_n[I_m,0]^{\mathrm{T}}$. If B is partitioned as

$$B=\begin{bmatrix}B_m, & 0\\ E, & F\end{bmatrix},$$

B_m being of order $m\times m$, then $a_{11}[I_m,0]B^{-1}B^{\mathrm{T}}[I_m,0]^{\mathrm{T}}=a_{11}B_m^{-1}B_m^{\mathrm{T}}$. Thus, again by Remark 1, $\hat{A}_m$ is centrogonal, $1\leqslant m\leqslant n-1$.

Necessity will be proved by induction on n. The cases $n = 1$ and $n = 2$ are easy to check. Let $n \geqslant 3$ and let A_{n+1} be partitioned as

$$A_{n+1} = \left(\begin{array}{c|ccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1} \\ \hline a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & a_{n+1,n+1} \end{array}\right) = \begin{bmatrix} \alpha_1, & \mathbf{b}_1^{\mathrm{T}}, & \alpha_2 \\ \mathbf{b}_2, & G, & \mathbf{b}_3 \\ \alpha_3, & \mathbf{b}_4^{\mathrm{T}}, & \alpha_4 \end{bmatrix}.$$

Since the 1×1 matrix (a_{11}) is centrogonal, it holds that $a_{11}^2 = \alpha_1^2 = 1$. Put

$$\mathbf{a}_1 = (a_{11}, \ldots, a_{1,n+1})^{\mathrm{T}},$$

$$\mathbf{a}_2 = (a_{11}, \ldots, a_{1n})^{\mathrm{T}},$$

$$\mathbf{a}_3 = (a_{11}, \ldots, a_{1,n-1})^{\mathrm{T}},$$

$$B_i = B(\mathbf{a}_i), \quad 1 \leqslant i \leqslant 3,\ J_{n-1} = J.$$

Then

$$B_1 = \begin{bmatrix} \alpha_1, & \mathbf{0}^{\mathrm{T}}, & 0 \\ \mathbf{b}_1, & B_3, & \mathbf{0} \\ \alpha_2, & \mathbf{b}_1^{\mathrm{T}} J, & \alpha_1 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} \alpha_1, & \mathbf{0}^{\mathrm{T}} \\ \mathbf{b}_1, & B_3 \end{bmatrix}.$$

By the induction hypothesis one has $B_2 A_n = \alpha_1 B_2^{\mathrm{T}}$, thus

$$\mathbf{b}_2 = -\alpha_1 B_3^{-1} \mathbf{b}_1, \tag{2.5}$$

$$B_3 G = \alpha_1 B_3^{\mathrm{T}} - \mathbf{b}_1 \mathbf{b}_1^{\mathrm{T}}. \tag{2.6}$$

Furthermore

$$A_{n+1}^{\mathrm{R}} = \begin{bmatrix} \alpha_4, & \mathbf{b}_4^{\mathrm{T}} J, & \alpha_3 \\ J\mathbf{b}_3, & JGJ, & J\mathbf{b}_2 \\ \alpha_2, & \mathbf{b}_1^{\mathrm{T}} J, & \alpha_1 \end{bmatrix}.$$

Let $A_{n+1}^{\mathrm{R}} A_{n+1} = I_{n+1}$ be partitioned accordingly. Therefore,

$$\alpha_1 \mathbf{b}_3 = -GJ\mathbf{b}_2 - \alpha_3 \mathbf{b}_2, \tag{2.7}$$

$$\mathbf{b}_1^{\mathrm{T}} J \mathbf{b}_2 = -\alpha_1(\alpha_2 + \alpha_3), \tag{2.8}$$

$$\alpha_2 \mathbf{b}_1^{\mathrm{T}} + \mathbf{b}_1^{\mathrm{T}} JG + \alpha_1 \mathbf{b}_4^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}, \tag{2.9}$$

$$\alpha_2^2 + \mathbf{b}_1^{\mathrm{T}} J \mathbf{b}_3 + \alpha_1 \alpha_4 = 1. \tag{2.10}$$

We have to show that $B_1 A_{n+1} = \alpha_1 B_1^{\mathrm{T}}$, or

$$\begin{bmatrix} 1, & \alpha_1 \mathbf{b}_1^{\mathrm{T}}, & \alpha_1\alpha_2 \\ \alpha_1 \mathbf{b}_1 + B_3 \mathbf{b}_2, & \mathbf{b}_1 \mathbf{b}_1^{\mathrm{T}} + B_3 G, & \alpha_2 \mathbf{b}_1 + B_3 \mathbf{b}_3 \\ \alpha_1\alpha_2 + \mathbf{b}_1^{\mathrm{T}} J \mathbf{b}_2 + \alpha_1\alpha_3, & \alpha_2 \mathbf{b}_1^{\mathrm{T}} + \mathbf{b}_1^{\mathrm{T}} J G + \alpha_1 \mathbf{b}_4^{\mathrm{T}}, & \alpha_2^2 + \mathbf{b}_1^{\mathrm{T}} J \mathbf{b}_3 + \alpha_1\alpha_4 \end{bmatrix}$$
$$= \alpha_1 \begin{bmatrix} \alpha_1, & \mathbf{b}_1^{\mathrm{T}}, & \alpha_2 \\ \mathbf{0}, & B_3^{\mathrm{T}}, & J\mathbf{b}_1 \\ 0, & \mathbf{0}^{\mathrm{T}}, & \alpha_1 \end{bmatrix}.$$

Observing (2.5), (2.6), (2.8)–(2.10), it remains to show that

$$\alpha_2 \mathbf{b}_1 + B_3 \mathbf{b}_3 = \alpha_1 J \mathbf{b}_1.$$

Applying (2.5) and (2.7) one gets

$$B_3 \mathbf{b}_3 = B_3 G J B_3^{-1} \mathbf{b}_1 + \alpha_3 \mathbf{b}_1.$$

From (2.6) and (2.8) one then gets

$$\begin{aligned} \alpha_2 \mathbf{b}_1 + B_3 \mathbf{b}_3 &= (\alpha_1 + \alpha_3)\mathbf{b}_1 + \alpha_1 B_3^{\mathrm{T}} J B_3^{-1} \mathbf{b}_1 - (\mathbf{b}_1^{\mathrm{T}} J B_3^{-1} \mathbf{b}_1)\mathbf{b}_1 \\ &= (\alpha_2 + \alpha_3)\mathbf{b}_1 + \alpha_1 B_3^{\mathrm{T}} J B_3^{-1} \mathbf{b}_1 + \alpha_1(\mathbf{b}_1^{\mathrm{T}} J \mathbf{b}_2)\mathbf{b}_1 \\ &= \alpha_1 B_3^{\mathrm{T}} J B_3^{-1} \mathbf{b}_1. \end{aligned}$$

Since $B_3^{\mathrm{T}} = J B_3 J$, the assertion follows. □

Let $\mathscr{A}_n$ be the set of all $n \times n$ matrices which are principally centrogonal. Then Theorem 2 has, for instance, the following consequences:

(i) Define for $A, A' \in \mathscr{A}_n$ a product by

$$A \circ A' = a_{11} a'_{11} (B(\mathbf{a}) B(\mathbf{a}'))^{-1} (B(\mathbf{a}) B(\mathbf{a}'))^{\mathrm{T}},$$

where $\mathbf{a}^{\mathrm{T}}$, $\mathbf{a}'^{\mathrm{T}}$ are the first rows of A and A', respectively. Since there is a uniquely determined $\mathbf{a}''$ such that $B(\mathbf{a})B(\mathbf{a}') = B(\mathbf{a}'')$ and $a''_{11} = a_{11} a'_{11}$, $\mathscr{A}_n$ is closed w.r.t. this product. It is easily shown that $(\mathscr{A}_n, \circ)$ is a commutative group with I_n as unit and A^{T} as inverse of A.

(ii) For $A \in \mathscr{A}_n$ one has $\det A = a_{11}^n$.

Remark 2. If $\|A\|$ is a matrix norm, a condition number $\kappa(A)$ for regular A is defined by

$$\kappa(A) = \|A\| \, \|A^{-1}\|,$$

cf. e.g. Golub and van Loan [3, p. 80]. For the commonly used norms such as

$$\|A\|_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |a_{ij}|, \ \|A\|_{\mathrm{F}} = (\operatorname{trace} AA^{\mathrm{T}})^{1/2}, \ \|A\|_\infty = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}|$$

or

$$\|A\|_S = \lambda_M^{1/2},$$

where λ_M is the largest eigenvalue of $A^{\mathrm{T}}A$, one can easily show that

$$\|A^{-1}\| = \|JAJ\| = \|A\|,$$

if A is centrogonal.

3. An exercise on binomial coefficients

In this section we will study the introductory example in view of Theorem 2. For that purpose let for $n \in \mathbb{N}$ the matrix $B_n = B = (b_{ij})$ be defined by

$$b_{ij} = (-1)^{i+j}\binom{n}{i-j}, \quad 1 \leqslant i, j \leqslant n. \tag{3.1}$$

It is easily seen that $B = B(\mathbf{a})$, where

$$\mathbf{a} = (a_1, \dots, a_n)^{\mathrm{T}}, \qquad a_i = (-1)^{i+1}\binom{n}{i-1}, \quad 1 \leqslant i \leqslant n,$$

and $B^{-1} = (b'_{ij})$ where

$$b'_{ij} = \binom{n-1+i-j}{n-1}, \quad 1 \leqslant i, j \leqslant n.$$

By Theorem 2 the matrix $A = \eta B^{-1}B^{\mathrm{T}}$, $\eta \in \{-1, 1\}$, is principally centrogonal. (For certain reasons later on we will choose $\eta = (-1)^{n+1}$.) For calculating the entries $a_{ij} = \eta \sum_{\varrho} b'_{i\varrho} b_{j\varrho}$ of A it turns out that, as it is often the case when dealing with binomial coefficients, a more general formula is easier to handle. Here the following is useful: For $n \in \mathbb{N}$, $k, \ell \in \mathbb{N}_0$, $n + \ell \geqslant k + 1$, it holds that

$$\sum_{\varrho=0}^{k} (-1)^{k-\varrho}\binom{n}{\varrho}\binom{n-1+\ell-k+\varrho}{n-1} = \binom{n+\ell}{k}\binom{n-1+\ell-k}{\ell}. \tag{3.2}$$

Eq. (3.2) can be proved by induction on k: Obviously, (3.2) holds for $k = 0$ or $\ell = 0$. Let now $k, \ell \in \mathbb{N}_0$, $n \in \mathbb{N}$, $n + \ell \geqslant k + 2$, and without loss of generality, $\ell \geqslant 1$. Then by the induction hypothesis

$$\sum_{\varrho=0}^{k+1} (-1)^{k-\varrho}\binom{n}{\varrho}\binom{n-1+\ell-1-k+\varrho}{n-1}$$
$$= \binom{n}{k+1}\binom{n-1+\ell}{n-1} - \binom{n-1+\ell}{k}\binom{n-1+\ell-1-k}{\ell-1},$$

which is easily seen to be zero if $n \leqslant k+1$. But in this case the right-hand side of (3.2) is zero, too. If $n > k+1$ a straightforward calculation yields

$$\begin{aligned}&\binom{n}{k+1}\binom{n-1+\ell}{n-1} - \binom{n-1+\ell}{k}\binom{n-1+\ell-1-k}{\ell-1}\\ &\quad= \binom{n+\ell}{k+1}\binom{n-1+\ell-1-k}{\ell}.\end{aligned}$$

Now for a_{ij} one gets

$$\begin{aligned}a_{ij} &= \eta(-1)^j \sum_{\mu=1}^{j}(-1)^\mu \binom{n-1+i-\mu}{n-1}\binom{n}{j-\mu}\\ &= \eta\sum_{\varrho=0}^{j-1}(-1)^\varrho \binom{n-1+i-j+\varrho}{n-1}\binom{n}{\varrho}.\end{aligned}$$

Putting in (3.2) $k = j-1$, $\ell = i-1$, one obtains

$$a_{ij} = \eta(-1)^{j-1}\binom{n-1+i}{j-1}\binom{n-1+i-j}{i-1}. \tag{3.3}$$

(For $\eta = 1$ and $n = 4$ this is the matrix (1.1).) From this result further identities can be derived, a direct proof of which appears to be formidable. For $n \in \mathbb{N}$, $1 \leqslant i, j \leqslant n$, $\eta = 1$ one gets from $BA = B^{\mathrm{T}}$ that

$$\sum_{\varrho=1}^{n}(-1)^{\varrho+1}\binom{n}{i-\varrho}\binom{n-1+\varrho}{j-1}\binom{n-1+\varrho-j}{\varrho-1} = \binom{n}{j-i}$$

and from $A^{\mathrm{R}}A = I$ that

$$\begin{aligned}&(-1)^{n+1+j}\sum_{\varrho=1}^{n}(-1)^\varrho\binom{2n-i}{n-\varrho}\binom{n-1-i+\varrho}{n-i}\\ &\quad\times\binom{n-1+\varrho}{j-1}\binom{n-1+\varrho-j}{\varrho-1} = \delta_{ij},\end{aligned}$$

where δ_{ij} is the Kronecker symbol.

Besides principal centrogonality the matrix A, when suitably normalized, shares another significant property with the unit matrix: It has the same characteristic polynomial. This will follow from the following proposition.

Proposition 1. *The matrix* $A = (a_{ij})$ *defined by* (3.3) *is similar to the matrix* $\eta(-1)^{n+1}B$, *where B is defined by* (3.1).

Proof. Take the matrix D and its inverse D^{-1} as in Example 1. We show that $A = \eta(-1)^{n+1}D^{-1}BD$. For $D^{-1}B = (s_{ij})$ one gets with formula (5.22) in [5] that

$$s_{ij} = (-1)^{n+j}\sum_{\varrho=1}^{n}\binom{i-1}{n-\varrho}\binom{n}{\varrho-j} = (-1)^{n+j}\binom{n-1+i}{n-j}.$$

For $D^{-1}BD = (t_{ij})$ one then obtains, applying formulas (5.21) and (5.16) in [5] that

$$\begin{aligned}
t_{ij} &= (-1)^{n+1+j}\sum_{\varrho=1}^{n}(-1)^{\varrho}\binom{n-1+i}{n-\varrho}\binom{n-\varrho}{j-1}\\
&= (-1)^{n+1+j}\sum_{\varrho=1}^{n-j+1}(-1)^{\varrho}\binom{n+i-1}{j-1}\binom{n+i-j}{\varrho+i-1}\\
&= (-1)^{n+i+j}\binom{n+i-1}{j-1}\sum_{\sigma=i}^{n+i-j}(-1)^{\sigma}\binom{n+i-j}{\sigma}\\
&= (-1)^{n+i+j}\binom{n+i-1}{j-1}\left[\sum_{\sigma=0}^{n+i-j}(-1)^{\sigma}\binom{n+i-j}{\sigma}\right.\\
&\quad\left.-\sum_{\sigma=0}^{i-1}(-1)^{\sigma}\binom{n+i-j}{\sigma}\right]\\
&= (-1)^{n+i+j+1}\binom{n+i-1}{j-1}\sum_{\sigma=0}^{i-1}(-1)^{\sigma}\binom{n+i-j}{\sigma}\\
&= (-1)^{n+j}\binom{n+i-1}{j-1}\binom{n+i-j-1}{i-1} = \eta(-1)^{n+1}a_{ij}. \qquad \square
\end{aligned}$$

Since B is a triangular matrix with all diagonal entries equal to 1, Proposition 1 implies that the characteristic polynomial of $\hat{A} = \eta(-1)^{n+1}A$ is

$$\det(\hat{A} - \lambda I) = \det(B - \lambda I) = (1-\lambda)^n,$$

in particular $\det \hat{A} = 1$ and trace $\hat{A} = n$. A further consequence is that

$$rg(\hat{A} - I) = rg(B - I) = n - 1,$$

i.e. the eigenspace of $\hat{A}$ corresponding to the eigenvalue $\lambda = 1$ has dimension 1. We leave it as an exercise to show that $\mathbf{x}_r = (1, 1, \ldots, 1)^{\mathrm{T}}$ and $\mathbf{x}_\ell^{\mathrm{T}} = (x_1, \ldots, x_n)$, where $x_i = (-1)^{i+1}\binom{n-1}{i-1}$, $1 \leqslant i \leqslant n$, are right and left eigenvectors of $\hat{A}$, respectively.

One may also note that A is highly ill-conditioned (cf. also Remark 2). Take for instance the smallest and the largest eigenvalue of AA^{T}, λ_m and λ_M, respectively. Writing $A^{\mathrm{T}} = [\mathbf{a}_1, \ldots, \mathbf{a}_n]^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ one has $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}_j = \delta_{ij}$. Hence one gets for $\hat{\mathbf{x}}_1 = \mathbf{x}_1/(\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_1)^{1/2}$ that

$$\lambda_m = \min\{\mathbf{x}^{\mathrm{T}} A A^{\mathrm{T}} \mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{x} = 1\} \leqslant \|A^{\mathrm{T}}\hat{\mathbf{x}}_1\|^2 = (\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_1)^{-1}.$$

Because of $(A^{\mathrm{T}})^{-1} = JA^{\mathrm{T}}J$ one obtains $\mathbf{x}_1^{\mathrm{T}} = (a_{n,n}, a_{n,n-1,\dots,}a_{n,1})$, thus

$$\lambda_m \leqslant \left\{ \sum_{j=1}^{n} \left[\binom{2n-1}{j-1} \binom{2n-1-j}{n-1} \right]^2 \right\}^{-1}.$$

Similarly, if $\mathbf{e}_n$ denotes the nth unit vector,

$$\begin{aligned} \lambda_M &= \max\left\{\mathbf{x}^{\mathrm{T}} A A^{\mathrm{T}} \mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{x} = 1\right\} \geqslant \|A^{\mathrm{T}}\mathbf{e}_n\|^2 \\ &= \sum_{j=1}^{n} \left[\binom{2n-1}{j-1} \binom{2n-1-j}{n-1} \right]^2. \end{aligned}$$

Acknowledgement

We wish to thank two anonymous referees for their critical reading and constructive comments and also L. Imhof for several discussions on this topic.

References

[1] A.C. Aitken, Determinants and Matrices, Oliver and Boyd, Edinburgh, 1956.
[2] R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York, 1960.
[3] G.H. Golub, C.F. van Loan, Matrix Computations, second ed., Johns Hopkins, University Press, Baltimore, 1989.
[4] I.J. Good, The inverse of a centrosymmetric matrix, Technometrics 12 (1970) 925–928.
[5] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics, second ed., Addison-Wesley, Reading, MA, 1994.
[6] R.D. Hill, S.R. Waters, On κ-real and κ-hermitian matrices, Linear Algebra Appl. 169 (1992) 17–29.
[7] T. Muir, A Treatise On the Theory of Determinants, Dover, New York, 1960.
[8] J.R. Weaver, Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, and eigenvectors, Amer. Math. Monthly 92 (1985) 711–717.